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A low-cost experiment to visualise the Fourier series: video analysis of a real plucked coiled spring

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Abstract
In the present work, we develop a low-cost and simple experiment to visualise Fourier’s synthesis using a short, soft, and light plastic coiled spring oscillating in a horizontal plane, and a basic camera (120 fps). It is shown that the spring obeys a linear wave differential equation, as gravitational influence is neglected. A nonlinear criterion is evaluated to determine if magnitudes of the parameters in the initial conditions satisfy the linear wave equation. Our setup promotes some desirable characteristics that make Fourier’s synthesis experiments feasible, visual, and enlightening: (i) it requires few, common, and cheap resources, and the experiment can be carried out even in a high-school laboratory; (ii) since the spring’s tension is small (∼1 N, on average), the frequencies of normal modes are low (close to 2 Hz), and therefore, it is possible to record the oscillations just with the camera and extract a considerable number of position and time data in just one cycle; (iii) when the video is loaded in the Tracker free software, it can be reproduced in slow motion. Since the frequencies involved are low, an interesting and instructive temporal sequence of images of the spring displaying the typical trapezoidal shape appears clearly; (iv) the tools associated with the Tracker software tools can yield the relevant oscillation parameters, such as the damping constant, amplitudes, frequencies, and phases; and (v) it is possible to carry out superposition of a snapshot of the spring in Tracker at any time, and to draw the related Fourier synthesis graphs. The visual match between the shape of the spring and the theoretical graph is remarkable, and can be enhanced by adding the damping term.

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Introduction

Some of the best-known physics textbooks for undergraduates [1–4] devote several pages to explaining standing waves in strings and pipes, but very little space to discussing Fourier’s synthesis and analysis. Therefore, students have to learn this important subject in both advanced calculus and mathematical physics books [5, 6]. Certainly, these mathematical contexts provide a good support for students to understand the formalism of Fourier’s series, and its use in the solution of many kinds of differential equations. Unfortunately, this very formal approach does not relate fully to real experiments, in which periodic, but not sinusoidal, harmonic waves are involved. Indeed, there many papers and laboratory guides exploring how to produce standing waves in strings, springs, or pipes, and explaining physics concepts about oscillations and their relations, for instance normal modes of vibration and even musical tones. Two recent examples of this predominant trend are a paper that employs a digital oscilloscope with the fast Fourier transform to analyse standing waves in guitar strings [7], and another paper that uses the free software Tracker [8, 9] for video analysis to study standing waves in a Slinky [10]. However, it is very difficult to find papers that suggest experiments to illustrate and discuss Fourier’s synthesis in an undergraduate laboratory.

Recently, two papers have been published that propose and investigate experiments on Fourier’s synthesis using the technique of video analysis [11, 12]. The first and more complex of the two [11] focuses on nonlinear effects in a string, and actually demands the use of an advanced physics laboratory. The second paper [12] addresses the vibrational motion of a plucked guitar string at two different locations along the string’s length, and with different initial amplitudes, using a high-speed camera (1200 fps). This clear paper is a significant contribution to the field because it compares a vibrating string to the linear wave equation, and obtains consistent results. For instance, in the case of small plucked amplitudes, the string shows a predictable shape obeying the linear wave equation, regardless of the pluck’s position. Nevertheless, strong nonlinear effects arise for larger plucked amplitudes. Another paper [13] comments on the conclusions in [12], further explaining and clarifying these nonlinear aspects and presenting a criterion that goes beyond the use of plucked amplitudes to discriminate between linear and nonlinear effects.

Although both experiments are highly relevant, they can hardly be performed in a standard undergraduate physics laboratory because the required equipment is generally not available in such laboratories. In the present work, we describe a low-cost and simple experiment to visualise the Fourier’s synthesis using a short, soft, and light plastic coiled spring, and a basic camera. All data are gathered through Tracker [8, 9], a free video analysis software, and analysed by the Origin [14] graphics software. Naturally, the spring is assumed to obey a linear wave differential equation, but considerable damping effects in the wave behaviour can be introduced manually into the model.

Our setup promotes some desirable characteristics that make a Fourier’s synthesis experiment feasible, visual, and enlightening. These are as follows. (i) The experiment requires common and cheap resources. Teachers can carry out the experiment even in a high-school laboratory. (ii) The spring tension is small (1 N on average), and thereby the frequencies of normal modes are low (close to 2 Hz). Therefore, it is possible to record the oscillations with a basic camera (120 fps), and extract a considerable number of position and
time data for just one cycle. A guitar string was chosen in [12], where tensions and frequencies were very high, demanding a high-speed, more expensive camera, roughly one order of magnitude greater (1200 fps). (iii) In the Tracker software, the video can be reproduced in slow motion. Since the frequencies involved are low, a clear, interesting and instructive sequence of images, typically trapezoidal shapes of the spring’s time evolution, appear. (iv) Tracker allows us to visually select and mark each point of the spring at different times, and automatically generate time versus position plots. Data can be exported to Origin in order to be analysed and fit to suggested theoretical models. In this way, it is easy to obtain the relevant spring oscillation parameters, such as the damping constant, amplitudes, frequencies, and phases. (v) It is possible to carry out the frequency analysis of a spring’s snapshot in Tracker at any time, with the related graph of the Fourier synthesis for a small number of Fourier components. The visual match between the spring profile and the theoretically-computed graph is remarkable, and can be enhanced by adding a damping term.

**Theoretical model**

The string vibrates on a plane after been touched or disturbed. $y = y(x, t)$ corresponds to the transversal displacement of the string at position $x$ and at time $t$, with $0 \leq x \leq L$ and $t \geq 0$ (see figure 1).

The wave equation neglecting the string weight and considering small transversal amplitudes is given by

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}, \tag{1}$$

where $\mu$ is the linear density of the string, in kg m$^{-1}$, and $T$ is the tension applied on the string ends when no waves are present, in Newtons; both are considered constant. It is usual to write the wave equation in terms of the longitudinal velocity $v$, defined by the following:

$$v^2 = \frac{T}{\mu}. \tag{2}$$

Therefore, equation (1) can be written as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}. \tag{3}$$

The boundary conditions are as follows:

(a) The string is always locked at the edges: $y(0, t) = y(L, t) = 0 \quad \forall \ t \geq 0$.
(b) The initial condition of the string is $y(x, 0) = f(x)$.
(c) The initial velocity of the string is given by $\frac{\partial y}{\partial t}(x, 0) = g(x).$
(a) and (b) imply that \( f(0) = f(L) = g(0) = g(L) = 0 \). In this paper, we consider the specific case in which the initial displacement \( f(x) = 0 \) and the initial velocity is zero \( g(x) = 0 \). In other words, the string is displaced from its rest position, held for a while, and then released at \( t = 0 \).

In this paper, we are considering that the coiled spring has length \( L \), and is plucked at an arbitrary point \( x = \epsilon L; 0 < \epsilon \leq 1/2 \) until the position \( y = -h \); \( L \) and \( h \) are positive constants. The well-known solution of equation (3), as well as the specific piecewise functions problem treated in this paper, can be found in appendix A.

When \( \epsilon = 1/2 \), the spring is plucked at the middle point until the position \( y = -h \) (figure 2).

Using equation (A.3) to calculate the coefficients of the sine Fourier series, we have

\[
k_n = -\frac{8h}{n\pi^2} \sin\left(\frac{n\pi}{2}\right).
\]

If \( n \) is even, we have \( k_n = 0 \), remaining only those terms with \( n \) odd. Writing \( n = 2m + 1 \) in equation (A.4), the solution becomes

\[
y(x, t) = \frac{8h}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(2m + 1)^2} \sin\left(\frac{(2m + 1)\pi x}{L}\right) \cos\left(\frac{(2m + 1)\pi vt}{L}\right).
\]

The first five terms are

\[
y(x, t) = -\frac{8h}{\pi^2} \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi vt}{L}\right) + \frac{8h}{9\pi^2} \sin\left(\frac{3\pi x}{L}\right) \cos\left(\frac{3\pi vt}{L}\right) - \frac{8h}{25\pi^2} \sin\left(\frac{5\pi x}{L}\right) \cos\left(\frac{5\pi vt}{L}\right) + \frac{8h}{49\pi^2} \sin\left(\frac{7\pi x}{L}\right) \cos\left(\frac{7\pi vt}{L}\right)
\]

\[
- \frac{8h}{81\pi^2} \sin\left(\frac{9\pi x}{L}\right) \cos\left(\frac{9\pi vt}{L}\right) + \ldots
\]

It should be noted that the linear equation (1) does not fully describe the shape of the spring as a function of time. This is due to the fact that this model does not take into account large amplitudes. Rowland [13] presents the Kirchhoff–Carrier nonlinear differential wave equation to address large-amplitude transverse waves. According to reference [13], to remove the complications arising from transverse–longitudinal mode coupling, this equation assumes that tension \( T \) remains uniform across the string during oscillations. Transverse motion in one plane only is given by

\[
\mu \frac{\partial^2 y}{\partial t^2} = T [1 + N(x)] \frac{\partial^2 y}{\partial x^2}.
\]
The extra term \( N(x) \) depends on \( \frac{\partial^2 u}{\partial y^2} \) and also on the dimensionless constant \( e = (L - L_r)/L_r \), where \( L_r \) corresponds to the spring’s length under zero tension (relaxed spring). Note that the constant \( e \) gives the relative measurement of the spring.

Rowland establishes a boundary between the linear and nonlinear behaviour in terms of the key physical parameters. When \( (k_1/L)^2/e \) is much smaller than one, one can consider the linear behaviour \([13]\). Taking into account the value of \( k_1 \) obtained from equation (4), the boundary parameter becomes

\[
\frac{64k_1^2}{\pi^4L^2e}.
\]

Results analysis

The experimental setup, presented in figure 3 (upper picture), consists of a plastic toy coiled spring (29.14 g, 53 mm in diameter, and 0.06 m in length under zero tension) fixed by two dynamometers in the horizontal plane. The spring was extended to 2.50 m (tension about 0.9 N at rest), and then symmetrically plucked to create an amplitude of about 0.50 m. This coiled spring presents higher elasticity, and since its relative elasticity is
\[ e = (2.50 - 0.06)/0.06 = 40.67, \] its linear density is not constant. We used a powerful source of light, and a camera with a configuration of 120 fps and resolution frame of 640 × 480 pixels. The camera was positioned approximately 2 m from the ground where the experiment was performed.

In order to estimate the uncertainty of the coiled spring element position measurement, the uncertainty estimation method in a single measurement was used [15]. This is also known as Type B uncertainty, which can be obtained from an assumed probability density function based on the degree of belief that an event will occur, replacing the statistical treatment known as Type A uncertainty [16]. This procedure is implemented in cases of direct measurements, and is widely used in video analysis where the spot of the spring element is assumed as a region of probability [17]. Thus, instead of filming the experiment several times and doing numerous video analyses, we performed the experiment only once for an initial configuration, and estimated the position uncertainty within the region showing the spring element.

The estimated video analysis position uncertainty is approximately 0.01 m, while the correction due to the fifth term of equation (6) is 0.005 m. It is not necessary to calculate higher-order Fourier terms.

The upper picture in figure 3 presents a snapshot of the Tracker software and the video analysis for two different positions \( x = 1.25 \text{ m} \) and \( x = 2.00 \text{ m} \) (red central trace and blue right trace, respectively). It should be noted that the form of the time evolution graph (lower picture of figure 3) is similar to that corresponding to the snapshot of the spring. This can be understood by looking at equation (6), as the transversal position of the spring is described by a sine series depending on \( x \) when the time is fixed, or cosine series depending on \( t \) when the position \( x \) is fixed.

In order to compare the model calculations using equation (6) to our video analysis data, we need to include the decay of the amplitude of the coiled spring as energy is dissipated due to frictional forces as, for instance, air resistance and energy losses through the supports [12]. For the sake of simplicity, we use an unique exponential decay parameter \( \gamma \), that tries to take into account all of those frictional factors. Thus, the first five terms shown in equation (6) are multiplied by an exponential decay term, and the final expression can be written as follows:

\[ Y(x, t) = e^{-\gamma t}y(x, t). \] (9)

After setting \( L = 2.5 \text{ m} \), and \( h = 0.53 \text{ m} \), and fixing the position of an element of the spring at the centre \( (x = 1.25 \text{ m}) \), we obtained the velocity \( v \) and the exponential parameter \( \gamma \) by fitting the time oscillation data using equation (9) and the Origin graphics software [14], as presented in figure 4 (upper left graph). We obtained the exponential decay factor \( \gamma = 0.219(3) \text{ s}^{-1} \) and the velocity \( v = 9.737(3) \text{ m s}^{-1} \), which is in reasonable agreement with the approximated value obtained by the dynamometers (tension), ruler (length), and scale (mass) \( \sqrt{0.9/0.029/2.7} \sim 9 \text{ m s}^{-1} \). The lower graphs of figure 4 represent the element of the spring at \( x = 2.00 \text{ m} \), and the full line represents the behaviour predicted by the theoretical model, equation (9), with the same \( v \) and \( \gamma \) parameters obtained from the upper case.

Figure 5 presents the coiled spring evolution in nine different instants of time (each one represented four frames after the other, which corresponds to a time interval of 4/120 s). The open white circles represent equation (9) using only five terms of the Fourier series (equation (6)) for each fixed instant of time. These coefficients depend linearly on both the pluck \( h \) and the \( \cos \left( \frac{\pi x}{L} \right) \). The pluck and length were directly measured. The velocity and the damping factor were obtained by fitting, as described above (see figure 4).
The linear theoretical model fits well with the experiment, despite the larger pluck–length ratio $h/L \cong 0.5/2.5 = 1/5 = 20\%$. This can be explained by the expression (equation (8)) \( \frac{6h^3}{T^2} = 0.000 \, 65 \), which is much smaller than 1, as discussed in reference [13].

Conclusions

The main contribution of this work is to demonstrate the feasibility of performing an experiment to visualise the Fourier’s synthesis in an undergraduate laboratory. Indeed, all former papers on this subject address vibrating strings. Naturally, this choice entails the need to employ more sophisticated equipment, such as electronic oscillations generators and/or high-speed cameras. Instead, a light and soft plastic coiled spring can provide a similar and low-frequency version of a vibrating string that still satisfies the linear wave equation (see equation (8)). Accordingly, the experimental setup turns out to be much simpler and cheaper.

The video analysis of oscillations of a plucked coiled spring at two different positions reveals the same pattern already detected in strings. In fact, if the initial spring configuration is triangular, its temporal evolution shows the well-known trapezoidal shape throughout the cycle, except in the maximum amplitude points, where the spring recovers its initial triangular aspect. Since low frequencies are involved, the respective velocities of any point in the spring are small. Therefore, with just a basic camera (120 fps) it is possible to capture videos that are
good enough to carry out quantitative analysis and obtain precise values of relevant parameters, such as angular frequency, phase, and damping constant. These parameters are essential to fill the gaps in the theoretical model and hence mathematically simulate the shape of the sum of the first five terms of the Fourier’s synthesis.

Finally, the superposition between the snapshot of the actual coiled spring’s shapes for different times and the respective graphs generated by the Fourier’s sum is extremely instructive and well-fitted, even with few terms. This visual and remarkable match of theory and practice endorse the relevance of this subject, rather neglected in introductory physics books, but mainly present in laboratory guides.

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Appendix A

Using separation of variables \[5, 6\] it is not difficult to find the well-known solution

\[y(x, t) = \sum_{n=1}^{\infty} y_n(x, t),\]  
(A.1)

where \(y_n(x, t)\) corresponds to

\[y_n(x, t) = \sin\left(\frac{n\pi x}{L}\right)\left[c_n \sin\left(\frac{n\pi vt}{L}\right) + k_n \cos\left(\frac{n\pi vt}{L}\right)\right].\]  
(A.2)

The \(c_n\) and \(k_n\) coefficients should be determined from the initial position \(f(x)\) and velocity \(g(x) = 0\).

From \(y(x, 0) = f(x)\), we have

\[f(x) = y(x, 0) = \sum_{n=1}^{\infty} k_n \sin\left(\frac{n\pi x}{L}\right),\]

and the \(k_n\) coefficients are

\[k_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.\]  
(A.3)

Moreover, from \(\frac{\partial y}{\partial t}(x, 0) = g(x) = 0\) and using the same condition above, we have

\[g(x) = \frac{\partial y}{\partial t}(x, 0) = \sum_{n=1}^{\infty} c_n \frac{n\pi v}{L} \sin\left(\frac{n\pi x}{L}\right) = 0.\]

The condition above implies that the all \(c_n\) coefficients are zero.

The solution is given by

\[y(x, t) = \sum_{n=1}^{\infty} k_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L}\right).\]  
(A.4)

Consider that the string has a length \(L\) and is plucked at an arbitrary point \(x = \varepsilon L; 0 < \varepsilon \leq 1/2\) until the position \(y = h; L\) and \(h\) are positive constants (figure A1).

Based on figure A1 we can write the initial position as below:

\[y(x, 0) = f(x) = -\frac{h}{\varepsilon L} x \quad 0 \leq x \leq \varepsilon L\]

\[y(x, 0) = f(x) = \frac{h}{(1 - \varepsilon)L}(x - L) \quad \varepsilon L \leq x \leq L.\]
Using equation (A.3) to calculate the coefficients of the Fourier series, we obtain

\[ k_n = \frac{-2h}{\varepsilon(1 - \varepsilon)n^2\pi^2} \sin(n\pi\varepsilon). \]  \hspace{1cm} (A.5)

Here, the solution of equation (A.4) becomes

\[ y(x, t) = -\frac{2h}{\varepsilon(1 - \varepsilon)n^2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi\varepsilon}{L}\right) \cos\left(\frac{n\pi vt}{L}\right). \]  \hspace{1cm} (A.6)

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