Physical pendulum experiments to enhance the understanding of moments of inertia and simple harmonic motion

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Physical pendulum experiments to enhance the understanding of moments of inertia and simple harmonic motion

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Abstract
This paper describes a set of experiments aimed at overcoming some of the difficulties experienced by students learning about the topics of moments of inertia and simple harmonic motion, both of which are often perceived to be complex topics amongst students during their first-year university courses. By combining both subjects in a discussion about physical pendula, in which the oscillation time periods for the periodic motion of several objects (a tennis ball, a thin beam, a hoop and a solid disc) are measured and compared, students are able to understand both topics at a higher level and also experience the synergistic effect of combining two or more physics themes in order to accelerate their learning whilst simultaneously raising their motivation. Special attention is given to the ‘ball and stick’ pendulum in which a block of material (treated as a point mass) can be moved along a shaft to create a composite pendulum whose time period exhibits a minimum value at a certain separation between the block and the rotation axis.

Introduction
Two topics that students in the age group 16–19 find challenging, be they A-level students or Level 1 university students, are oscillations and the physics of rotation. The former is taught within most A-level specifications in the UK [1] in the form of simple harmonic motion (SHM) but now in a far more qualitative manner than in the 1980s, for example; at Level 1 in most university physics programmes worldwide, SHM is covered in the form of the simple (point mass) pendulum and the physical pendulum [2]. The latter requires an appreciation of the concept of moment of inertia, which again is perceived as a difficult subject to understand. The physics of rotation, more generally, is covered only at a very basic level within A-level syllabuses [3]; in the first year of university courses, the idea of moment of inertia is most commonly introduced initially in the context of point masses separated from an axis of rotation around which they are revolving. At a deeper level, this subject is developed in the context of the rotation of a mass distribution which itself defines the geometry (shape) of the object, and thus provides an appreciation of the fact that the moment of inertia, and therefore rotational kinetic energy, of an object is dependent on this mass distribution rather than simply its mass.
Here, a demonstration lecture is described in which the authors bring these two topics together in order to illustrate practically how the time period of oscillation of various objects (different mass distributions) can be accurately predicted from an understanding of the fundamental physics of the simple and the physical pendulum. From the basic measurement of the oscillation time period of a simple pendulum of length $L$ which consists of a tennis ball tethered to a length of string, the oscillation periods, $T$, for a swinging rod, hoop and solid disc are predicted. Subsequent measurement of the latter three values of $T$ are found to agree very well with those predicted values, so demonstrating the value of the physics learnt, and how a deeper, synergistic understanding of both topics can be gained.

**Background to the simple pendulum and the physical pendulum**

A laboratory simple pendulum can be made most easily using a length, $L$, of ordinary shop-bought string and a tennis ball. A triple knot tied at the end of the string is forced into a very small slit cut in the rubber of the tennis ball to form the pendulum ‘bob’. The free end of the string is tied to a clamp which forms the pivot or axis of rotation. Figure 1 shows a schematic diagram of the pendulum.

In the figure, the bob is shown with an angular displacement, $\theta$, from the vertical; the restoring force which causes the tennis ball to move towards its equilibrium position is given by $F_{\text{res}}$, where:

$$F_{\text{res}} = -mg \sin \theta \quad (1)$$

in which $m$ is the mass of the bob and $g$ is the acceleration due to gravity. From Newton’s second law, this restoring force causes the linear acceleration such that:

$$-mg \sin \theta = m \frac{d^2x}{dt^2}. \quad (2)$$

This is valid in the case when $\theta$ is small so that $\sin \theta \sim \theta$ and $x \sim L\theta$ and hence:

$$-g\theta = L \frac{d^2\theta}{dt^2}. \quad (3)$$

Finally, this equation can be compared to the general equation for SHM, $\frac{d^2\phi}{dt^2} = -\omega^2 \phi$, in which $\omega$ is the angular frequency given by $\omega = \frac{2\pi}{T_{\text{simple}}}$, where $T_{\text{simple}}$ is the oscillation time period of this simple pendulum, which can be expressed in relation to its length by:

$$T_{\text{simple}} = 2\pi \sqrt{\frac{L}{g}}. \quad (4)$$

Theoretically, this equation is valid when the pendulum bob is a point mass separated from the rotation axis by a distance, $L$. In practice, for lengths of $L$ greater than around 50 cm the tennis ball acts effectively as a point mass, in the context of this experiment. Educationally, it is important to emphasize that the mass itself of the pendulum bob does not appear in the expression for its time period. Students initially find this a difficult concept.

A physical pendulum in which the bob is now an object whose mass is distributed over a certain geometry to give the object its characteristic shape can be analysed in a similar manner, as supported by figure 2. Here, the restoring torque, $\tau$, can be written as:

$$\tau = -mg \sin \theta d \quad (5)$$

in which the distance $d$ is shown in figure 2. For
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small angle oscillations, equation (5) reduces to:
\[ \tau = -(mgd)\theta. \]  

From elementary rotational kinematics, the torque causes the angular acceleration via \( \tau = I\alpha \), where \( I \) is the moment of inertia of the oscillating object and \( \alpha \) is the angular acceleration given by \( \alpha = \frac{d^2\theta}{dt^2} \). Thus, the equation for SHM of the physical pendulum can be written as:
\[ I\frac{d^2\theta}{dt^2} = -(mgd)\theta \]  
from which the oscillation period for the object whose mass is distributed over its own particular geometry or shape is:
\[ T_{\text{physical}} = 2\pi \sqrt{\frac{I}{mgd}}. \]  

In equation (8), the mass \( m \) is the total mass of the object and \( I \) is the total moment of inertia of the object. Generally, the moment of inertia is defined as \( I = \int r^2 \, dM \). Reference to texts such as [2] provides the moments of inertia expressions required for the four objects discussed in this paper.

Furthermore, if the oscillating body is composed of two or more contributing masses (for example, a metal rod and a metal block through which the rod was placed), then both masses contribute to the total mass, \( m \), and both moments of inertia would contribute to \( I \). This will become important later in the paper.

The case of the simple pendulum can be seen to fit the general, physical pendulum analysis. If the tennis ball is treated as a point mass, then its moment of inertia, \( I_{\text{ball}} \), is simply \( mL^2 \), but as the pendulum length is also the distance between the tennis ball centre and the rotation axis (pivot), \( L = d \), and so \( I_{\text{ball}} = md^2 \). Thus, this leads back to the formula for the oscillation time period of the ball, \( T_{\text{ball}} \), which is that of the simple pendulum:
\[ T_{\text{ball}} = 2\pi \sqrt{\frac{md^2}{mgd}} = 2\pi \sqrt{\frac{d}{g}} = 2\pi \sqrt{\frac{L}{g}}. \]  

Furthermore, equation (8) allows the oscillation time period to be predicted for an oscillating object of any shape, provided its moment of inertia is known. From the physics education point of view, this facilitates an excellent way of linking together two topics (moment of inertia and SHM) with a view to providing a less compartmentalized learning experience, and also the way in which mathematics—albeit at a simple level initially—is used as a tool for modelling real physical behaviour that can be observed with the intention of putting theory to the test, as will become evident later.

Oscillation time period measurements for objects of different shapes
Most commonly, 16–19 year old students experience as a classroom demonstration only the oscillation of a simple pendulum. Most will measure the time taken for several complete oscillations to occur in order to obtain an accurate value for \( T_{\text{simple}} \). From this value, hopefully they are encouraged to use equation (4) to calculate a value for the acceleration due to gravity for the Earth (9.81 m s\(^{-2}\)) [4].

However, with very little additional expense and effort, this learning experience can be extended and enhanced dramatically.
**Hoop or ring**

A circular aluminum hoop made with an aluminum rod of diameter \( \sim 8 \text{ mm} \) is the first geometry discussed in which the mass of the object is distributed over a well-defined region of space. The diameter of the hoop is chosen to be the same as the length of the simple pendulum comprising the tennis ball for reasons which will become clear shortly, and is considerably larger than the aluminum rod diameter. The moment of inertia of a hoop when rotating about an axis through its centre is simply \( I = \frac{1}{2} mR^2 \), where \( R \) is its radius. From a teaching point of view, this could either be accepted or derived \([5]\). However, when the hoop is hung from the clamp and set to oscillate, the axis of rotation is at one edge of the hoop rather than at its centre, and so the parallel axis theorem (PAT) \([6]\) is required to account for this displacement, leading to a moment of inertia, \( I_{\text{hoop}} \), given by:

\[
I_{\text{hoop}} = mR^2 + md^2 = 2mR^2 \quad (10)
\]

Since the distance between the centre of mass and the rotation axis, \( d \), is also equal to the radius of the hoop, \( R \). Thus, substitution of equation (10) into (8) yields the time period of oscillation of the hoop, given by:

\[
T_{\text{hoop}} = 2\pi \sqrt{\frac{2mR^2}{mgR}} = 2\pi \sqrt{\frac{2R}{g}}. \quad (11)
\]

Since the diameter, \( 2R \), of the hoop has been chosen to be equal to the length of the simple pendulum consisting of the string and tennis ball, the oscillation time period of the hoop is expected to be equal to \( T_{\text{ball}} \). Thus, the theory makes this first prediction, \( T_{\text{hoop}} = T_{\text{ball}} \) which is verified experimentally later.

**Solid disc**

A solid circular disc of radius \( R \) made from plywood forms the second object whose mass is distributed over a region of space. The exact material from which the disc is made is not critical since again its mass itself will not appear in the final equation for the time period of oscillation. The moment of inertia of a solid disc about an axis through its centre is \( I = \frac{1}{2} mR^2 \) \([7]\), but of course when suspended from the clamp its rotation axis is displaced a distance \( d = R \) from the centre of mass, so that its moment of inertia then is obtained using the PAT as:

\[
I_{\text{disc}} = \frac{1}{2} mR^2 + md^2 = \frac{3}{2} mR^2. \quad (12)
\]

Substitution of equation (13) into (8) provides the time period of the oscillating disc:

\[
T_{\text{disc}} = 2\pi \sqrt{\frac{(3/2) mR^2}{mgR}} = 2\pi \sqrt{\frac{(3/2) R}{g}}. \quad (13)
\]

Since the radius of the disc, \( R = L/2 \) (where \( L \) is the length of the simple pendulum comprising the tennis ball), the relation between the value of \( T_{\text{disc}} \) and \( T_{\text{ball}} \) is simply:

\[
T_{\text{disc}} = \sqrt{3/4} T_{\text{ball}}. \quad (14)
\]

Thus, the time period of the disc is predicted theoretically to be 86.6% of that of the tennis ball pendulum.

**Solid beam**

A solid beam, whose length is the same as the length of the tennis ball pendulum, is suspended from one end via a small hole drilled as close to the end as possible. This forms the third distributed mass in this investigation. A similar analysis to the hoop and the disc leads to a moment of inertia of \( \frac{1}{4} mL^2 \) \([8]\). Substitution into equation (8) leads to the time period for the oscillating beam:

\[
T_{\text{beam}} = 2\pi \sqrt{\frac{(1/3) mL^2}{mg(L/2)}} = 2\pi \sqrt{\frac{(2/3) L}{g}}. \quad (15)
\]

In terms of the time period of the tennis ball pendulum, the time period of the beam is given by:

\[
T_{\text{beam}} = \sqrt{2/3} T_{\text{ball}}. \quad (16)
\]

Thus, the time period of the beam is predicted theoretically to be 81.6% of that of the tennis ball pendulum.

**Measurements**

The analysis above allows the time periods of all four pendula to be predicted according to:

\[
T_{\text{ball}} = T_{\text{hoop}} = 1.155T_{\text{disc}} = 1.225T_{\text{beam}}. \quad (17)
\]

It is now necessary to measure experimentally the time periods and compare them to the
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Table 1. Oscillation time period measurements for the tennis ball, hoop, disc and beam pendula.

<table>
<thead>
<tr>
<th>Pendulum</th>
<th>(T_{30}) (s)</th>
<th>(T) (s)</th>
<th>(T_{\text{ball}}/T_{\text{shape}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tennis ball</td>
<td>52.88</td>
<td>1.76</td>
<td>1.00</td>
</tr>
<tr>
<td>Hoop</td>
<td>52.63</td>
<td>1.75</td>
<td>0.99</td>
</tr>
<tr>
<td>Disc</td>
<td>45.78</td>
<td>1.53</td>
<td>1.16</td>
</tr>
<tr>
<td>Beam</td>
<td>43.20</td>
<td>1.44</td>
<td>1.22</td>
</tr>
</tbody>
</table>

Theoretically predicted values. This is done using a regular laboratory stopclock to time \(~20–30\) full swings of each pendulum—students should be encouraged to do this \(~3\) times and to calculate average values. The length of the pendulum used (tennis ball and string) was 0.77 m; thus, a theoretical time period of 1.760 s is predicted. Table 1 shows the experimental data collected from such an investigation and confirms that the relationships between the time periods for each pendulum are consistent with the theoretical predictions to an accuracy of \(<2\%\). The final column in the table gives the ratio \(T_{\text{ball}}/T_{\text{shape}}\), where \(T_{\text{shape}}\) is the time period for each of the four objects used to make the pendulum. These measured values are in close agreement with those seen in equation (17) above, as expected.

Having investigated these four mass distributions, students should grasp the fundamental concepts that (a) the time period of oscillation for a pendulum undergoing SHM depends on its moment of inertia and (b) the moment of inertia itself depends on the shape or geometry of the object over which its mass is distributed. Beyond this, the use of simple mathematical substitution is emphasized as an important tool in understanding physical principles and physical observations. This latter aspect is taking to a much higher level in the section ‘Time period modelling for the ball and stick pendulum’.

Time period modelling for the ball and stick pendulum

Development of time period expression

The basic understanding of the oscillation behaviour of simple objects, as described above, can be developed significantly if a pendulum is made up of more than one object. Figure 3 shows a photograph and a schematic diagram of a ball and stick pendulum in which the shaft of the pendulum is made of a 9 mm diameter wooden dowel (the ‘stick’) onto which is threaded a wooden block (the ‘ball’), which can be positioned anywhere along the length of the shaft.

The oscillations of the shaft are governed by the fact that its moment of inertia is equivalent to the beam described above \((I_{\text{stick}} = \frac{1}{3}\mu L^2)\), where \(\mu\) is the mass of the dowel shaft and \(L\) is its length. However, the oscillations of the wooden block can be modelled in the same way that the tennis ball (point mass) was modelled earlier \((I_{\text{block}} = Ml^2)\), where \(M\) is the mass of the block and \(l\) is the distance between the centre of mass of the block and the axis of rotation (pivot).

Thus, the total moment of inertia that is required for substitution into equation (8) in order to obtain the time period for the ball and stick pendulum is \(I_{\text{total}}\), where

\[
I_{\text{total}} = Ml^2 + \frac{\mu L^2}{3}. \tag{18}
\]
Combining equations (8), (18) and (20) leads to an expression for the time period of the ball and stick pendulum in which the ball is placed at a distance $l$ from the pivot, given by:

$$T_{bs} = 2\pi \sqrt{\frac{Ml^2 + \mu l^2}{g(Ml + \mu l)}}.$$  \hspace{1cm} (21)

Again, it is now important to test this equation for specific conditions. Firstly, if the stick is ignored (i.e. the value of $\mu$ is set to zero) then for a value $l = L$, the time period is the same as that for a simple (point mass) pendulum as expected. Secondly, if the ball is ignored (i.e. $M = 0$) then for $l = L$, the time period is that found for the beam, $T_{beam} = 2\pi \sqrt{\frac{L^3}{3gL}}$. Finally, if the masses of the ball and stick are equal and the ball is positioned at $l = \frac{L}{2}$, the time period is given by:

$$T_{bs} = 2\pi \sqrt{\frac{M\left(\frac{L}{2}\right)^2 + \mu L^2}{g(M\left(\frac{L}{2}\right) + \frac{\mu L}{2})}} = 2\pi \sqrt{\frac{\mu L^2}{g\mu L}} \frac{\frac{7L}{12}}{g}.$$ \hspace{1cm} (22)

Thus, with suitable apparatus ($M = m$, $l = L$) it can be verified that the time period of the ball and stick pendulum under these conditions is a factor of 0.764 times that of a simple pendulum of the same length.

**Measurements and results**

Experimentally, the time period $T_{bs}$ is measured as a function of the distance between the pivot and the ball, $l$. Figure 5 shows a plot of $T_{bs}$ versus $l$ for measured data and also the theoretical curve based on equation (21) using the actual values of $M$ and $\mu$ for the real pendulum. The $T_{bs}$ value is obtained by timing the duration of 30 complete oscillations of the pendulum for each value of $l$. It is clear from the plot that a minimum in $T_{bs}$ occurs at a specific value of $l$. The realization of this feature would typically not be immediately obvious qualitatively if students were asked to predict the behaviour of the pendulum prior to any mathematical analysis. Furthermore, few students would recognize this minimum upon simple inspection (without further analysis) of equation (21), yet it is very clear in the experimental measurements.

This provides the exciting opportunity here to demonstrate to students the role of calculus [9].
A constant $\gamma = \frac{2\pi}{\sqrt{g}}$ can be introduced as can the functions $p(l)$ and $q(l)$, where $p(l) = (Ml^2 + \alpha)^{1/2}$ and $q(l) = (Ml + \beta)^{1/2}$, allowing the following expression to be written:

$$T_{bs} = \gamma \frac{p(l)}{q(l)}. \quad (25)$$

The quotient $p(l)/q(l)$ can be differentiated using the quotient rule for differentiation such that the first derivative of $T_{bs}$ is:

$$\frac{dT_{bs}}{dl} = \gamma \left\{ \frac{q(l)p(l) - p(l)q(l)}{[q(l)]^2} \right\}. \quad (26)$$

It can now be seen that the condition $\frac{dT_{bs}}{dl} = 0$ is met provided $q(l)p(l) - p(l)q(l) = 0$ and thus there is no need to evaluate every term in equation (26). The derivatives $p'(l)$ and $q'(l)$ are given by:

$$p'(l) = \frac{1}{2}(Ml^2 + \alpha)^{-1/2}2Ml = \frac{Ml}{(Ml^2 + \alpha)^{1/2}}$$

$$q'(l) = \frac{1}{2}(Ml + \beta)^{-1/2}M = \frac{M}{2(Ml + \beta)^{1/2}}. \quad (27)$$

Thus:

$$q(l)p'(l) - p(l)q'(l) = \frac{(Ml + \beta)^{1/2}Ml}{(Ml^2 + \alpha)^{1/2}} - \frac{(Ml^2 + \alpha)^{1/2}M}{2(Ml + \beta)^{1/2}} \quad (28)$$

and for the condition $q(l)p'(l) - p(l)q'(l) = 0$, it can be seen that:

$$(Ml + \beta)2l = (Ml^2 + \alpha). \quad (29)$$

Rearranging (29) yields a simple quadratic equation $Ml^2 + 2\beta l - \alpha = 0$ which can be solved for $l$ as below:

$$l = \frac{-2\beta \pm \sqrt{4\beta^2 + 4M\alpha}}{2M}. \quad (30)$$

Only the positive root is meaningful in the context of this experiment given by $l_+$, where:

$$l_+ = \frac{-\beta + \sqrt{\beta^2 + M\alpha}}{M}. \quad (31)$$

Again, this equation can be tested; firstly, the value of $l_+$ can be confirmed using the particular values of $M$ and $\mu$ for the actual pendulum used. However, the mass of the ball can be changed

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![Image of graph showing the relationship between the oscillation time period and the distance between the pivot and the ball for the "ball and stick" pendulum. The theoretical curve is shown with blue diamonds, and the experimentally measured values are shown with red squares.](image)

**Figure 5.** Plot of $T_{bs}$ versus $l$ for the ‘ball and stick’ pendulum. The theoretical curve predicted from equation (21) is shown as blue diamonds, and the experimentally measured values are shown as red squares.

in (a) identifying the existence of a turning point in the $T_{bs}$ versus $l$ curve, (b) finding the exact position of the ball needed to observe the minimum value of $T_{bs}$ and (c) thus demonstrating the power of using mathematics as a tool for aiding the understanding of the physics.

**The role of calculus**

In order to make equation (21) slightly more manageable, firstly the constants $\alpha$ and $\beta$ are introduced, where $\alpha = \frac{uL^2}{g}$ and $\beta = \frac{uL}{g}$, thus allowing equation (21) to be written as:

$$T_{bs} = 2\pi \sqrt{\frac{Ml^2 + \alpha}{g(Ml + \beta)}}, \quad (23)$$

Thus, in order to find the minimum value of $T_{bs}$, its first derivative with respect to $l$, $dT/dl$, must be found. Solving $dT/dl = 0$ will generate the value of $l$ for which $T_{bs}$ takes its minimum value.

Some simplification to the analysis can be made here since equation (23) can be recast as:

$$T_{bs} = \frac{2\pi}{g^{1/2}} \frac{(Ml^2 + \alpha)^{1/2}}{(Ml + \beta)^{1/2}}. \quad (24)$$
simply by using different materials to make the ball. In this way, the value of $l_\perp$ can be predicted theoretically for different values of $M$ and then confirmed experimentally. For the ball and stick pendulum used in the development of this paper ($L = 1.0 \, \text{m}$), $M = 0.061 \, \text{kg}$ and $\mu = 0.025 \, \text{kg}$, thus $\alpha = 0.0088 \, \text{kg}\, \text{m}^2$ and $\beta = 0.0128 \, \text{kg} \, \text{m}$. Equation (31) predicts a value of $l_\perp$ of 0.223 m; inspection of figure 5 shows that experimentally the value obtained is $0.22 \pm 0.01$ m in very good agreement with theory. Figure 6 shows how $l_\perp$ is expected to vary as a function of the mass of the ball for a value of stick mass of $\mu = 0.025 \, \text{kg}$. Several different balls were made using materials of different density so that a few experimental data points could be added to illustrate the excellent fit between experiment and theory that can be seen in the plot.

Summary

These experiments are useful in demonstrating (a) how basic theory can be developed to explain real observations made by the physicist, (b) the importance of basic mathematical tools in supporting the development of the physical theory and (c) the fact that physics education at this level does not need to be reliant on expensive apparatus. This latter point emphasizes that the programme of experiments in this paper could be followed in any country, however rich or poor, throughout the world, and, as a result, students can expect to acquire several skills important to theory development, experiment design, measurement and analysis.

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References


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